Will Heuristics Enhance the Success of Mathematics Problem Solving?

by

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ABSTRACT

The word heuristics, which has the same meaning as strategy, was given its 'taken-as-shared' meaning from the 1940s. Heuristics have four common features: (1) Heuristics are rules or methods to plan for solutions. (2) These rules or methods are plausible in nature. (3) They won't guarantee success in finding solutions. (4) They are hard to teach. The most common heuristics used in mathematics are trial and error, draw a diagram, construct a table, look for patterns, test and simulate, try a simpler problem, analogy and work backwards. There is a high success rate in mathematics problem solving for students engaging heuristics in their problem-solving process. The high correlation between devising a plan and the use of heuristics indicates the importance of employing heuristics while planning a solution for a mathematics problem. Teachers should illustrate the importance of heuristics to students in problem-solving activities. By doing so, students will assimilate these heuristics as their stock of tools which can be readily called into play at any time when rules and algorithms fail to tackle a problem.

INTRODUCTION

Twenty-first century mathematics education is about facing novel real-world problems, nurturing creative thinking skills and cultivating productive ways of learning. In attempting to innovate teaching and learning in order to prepare a new generation for the demands of this new era, many educators have embarked on researching for new teaching methods. The spoon-feeding method of yesterday will not suffice for this new era.

Many research findings have been transformed to good practices to be implemented in schools. However, mathematics still seems to be one of the difficult subjects for school students. Von Glaserfeld (1995) says that '[Educators] have noticed that many students were quite able to learn the necessary formulas and apply them to the limited range of textbook and test situations, but when faced with novel problems, they fell short and showed that they were far from having understood the relevant concepts and conceptual relations (p. 20)'. Lau et al (2003) find that the problem-solving ability of students decline drastically as the level of difficulty of mathematics problem such as real-world problems increases.

A study entitled 'Successful mathematics problem solving using heuristics' has been started in July 2004 to investigate the potentials of heuristics to deliver the emphases of mathematics education of the 21st century. It is hoped that the findings will contribute to developing innovative methods of teaching and learning of mathematics and hence to improving the performance of mathematics of our students.

This paper not only presents the theoretical rationale and the importance of heuristics to the teaching and learning process of mathematics, also reports the findings of the pilot study to date.

LITERATURE REVIEW

As mathematics educators, we operate with many assumptions. Ernest (1994) proposes the following assumptions:

(1) What assumption, whether conscious or unconscious from the teachers and students, underpin the teaching and learning of mathematics?
(2) Which teaching and learning theories are assumed?
(3) What means are adopted to achieve the aims of mathematics education?
(4) Are the ends and means consistent?

Cockroft (1982) points out that mathematics consists of three important elements. These are: facts and skills, conceptual structures, and general strategies and appreciation. Teaching mathematics must bring about a balance in learning among the three elements. The general strategies and appreciation plays a very important role in the mathematical process experienced by students while solving mathematics problems. One of the general strategies under this element is heuristic. This section explores the heuristics state of the art. It starts with a few definitions of heuristics, follows by the lists of heuristics and their applications and ends with the importance of heuristics to mathematics education.

What are heuristics?

In the words of Pólya (1973): 'Heuristics, or heuretic, or "ars inveniendi" was the name of a certain branch of study, not very clearly circumscribed, belonging to logic, or to philosophy, or to psychology, often outlined, seldom presented in detail, and as good as forgotten today. The aim of heuristics is to study the methods and rules of discovery and invention.' He further elaborates: 'I wish to call heuristic the study ... of means and methods of problem solving ... to entice the reader to do problems and to think about the means and methods he uses in doing them' (p. 112).

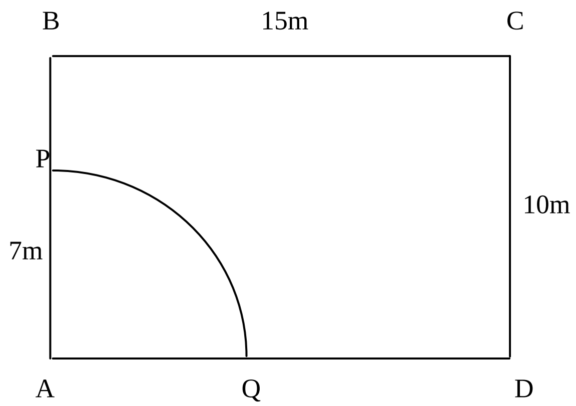
Many educators share his views on heuristics. Gelernter & Rochester (1958) postulate: '... a heuristic method (or a heuristic, to use the noun form) is a procedure that may lead us by a short cut to the goal we seek or it may lead us down a blind alley. It is impossible to predict the end result until the heuristic has been applied and the result checked by some formal process of reasoning. If the method does not have the characteristic that it may lead us astray, we would not call it a heuristic, but rather an algorithm.' (p. 337)

To Schoenfeld (1980), heuristic means: '... a general suggestion or strategy, independent of any particular topic or subject matter, that helps problem solvers approach and understand a problem and efficiently marshal their resources to solve it' (p. 9). In all the above cases, there are at least four common features of heuristics:

(1) Heuristics are rules or methods to plan for solutions.
(2) These rules or methods are plausible in nature.
(3) They won't guarantee success in finding solutions.
(4) They are hard to teach.

That is why Wilson (1967) comments: 'A heuristic is a decision mechanism, a way of behaving, which usually leads to desired outcomes, but with no guarantee of success. It is plausible in nature, giving guidance in the discovery of a solution.' (p. 3)

For instance, consider the following problem: A rectangular pasture has sides 15 m by 10 m. A cow is tied to one of the corners by a rope of length 7 m. Find the area of the field not being grazed by the cow.



In solving this problem, one must draw the above picture based on the information given. Drawing a picture is the heuristic that can be relied on to plan for a solution. However, the picture will only be correct if the problem solver sees PQ to be the arc of a circle of radius 7 m. However, drawing the correct picture will not guarantee the problem solver a solution if he or she does not know how to find the area of PBCDQ.

Consider another problem: Find x, satisfying the equation $x^4 - 13x^2 + 36 = 0$. Since $x^4 = (x^2)^2$, and introducing $y = x^2$, the auxiliary problem $y^2 - 13y + 36 = 0$ is obtained. In this case the notation $y = x^2$ and the auxiliary problem are the heuristics. The notation transforms the original problem to an intermediate problem that can help in finding a solution. To formulate this auxiliary problem and to conceive it as a means to solve the original problem is an achievement of intelligence. However, this auxiliary problem will not guarantee a solution if the problem solver does not know how to solve it for y and transform y back to x.

Heuristics lists

What actually are heuristics? Pólya (1973) presents a long list of heuristics in his 'Short Dictionary of Heuristics'.

Analogy	Problem to find, problem to prove
Auxiliary elements	Progress & achievement
Auxiliary problem	Reuctio, absurdum & indirect proof
Bright idea	Routine problem
Corollary	Rules of discovery
Decomposing & recombining	Rules of style
Definitions	Rules of teaching
Determination, hope & success	Setting up equations
Diagnosis	Signs of progress
Examine your guess	Specialization
Figures	Subconscious work
Generalization	Symmetry
Heuristic	Test by dimension
Heuristic reasoning	Wisdom of proverbs
Induction	Practical problems
Inventor's paradox	Variations
Lemma	Working backwards
Modern heuristics	Pedantry & mastery.

The following list, proposed by Begg (1994) is more applicable to school mathematics: guess and check, make a list, draw a picture, table or graph, find a pattern, a relationship, and/or a rule, make a model, solve a simpler problem first, work backwards, eliminate possibilities, try extreme cases, write a number sentence, act out a problem, restate a problem, check for hidden assumptions, change the point of view, and recognize when procedures are appropriate.

There is a close relationship between the heuristics used and the students' intellectual levels. Holton, Spicer, Thomas and Young (1996) suggest that primary students could independently and comfortably use: Draw a picture; Act it out; Use equipment; Guess and check. In the intermediate classes, students use a more extensive range of strategies: Act it out; Draw a diagram; Guess and check; Make a list or table; Work backwards; Work systematically; Look at patterns; Try a simpler case.

In Malaysia, eight different heuristics has been emphasized to students taking Additional Mathematics at the ordinary level. These are: trial and error, draw a diagram, make a table, look for patterns, test and simulate, try a simpler problem, analogy and work backwards. We will now provide illustrations in addition to the two examples in the last page to show how some of these heuristics can be used as a means to solve the original problem.

Trial and error

The problem solver starts with a trial and checks the feasibility of the answer obtained. This process is repeated until a correct answer is acquired. However, the choice of the trial should not be done randomly. Every subsequent choice should be made based on the previous trial.

That is to say, a better choice can be obtained by examining the previous trial with the information given in the problem. For instance, consider the following problem: A farmer rears some chickens and goats in his farm. There are a total of 50 heads and 140 legs. Find the number of chickens and the number of goats kept by this farmer. A problem solver can use trial and error to solve this problem. One can choose 25 chickens and 25 goats for a start. These answers will yield ($2 \times 25 + 4 \times 25$) or 150 legs which is 10 more legs than the information given in the problem. Since a goat has two more legs than a chicken, there should be more chickens than goats in this farm. Hence, the next better choice can be 28 chickens and 22 goats. This process is repeated until the correct answers are obtained.

Make a table

This heuristics is useful for solving problems that cannot be transformed into mathematical models. All possible cases for a problem will be listed in a table. However, a problem solver still needs other formulas or theorems to successfully solve it. The following example can be easily solved by making a table of it: Two fair dices are tossed one after another. Find the probability that the outcomes have a sum of 9.

(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)
(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)

The outcomes having a sum of 9 are (3,6), (4,5), (5,4) and (6,3).
The number of such outcomes = 4.
The total number of outcomes = 36.
$\therefore$ the probability of getting an outcome having a sum of 9 = 4/36 = 1/9

Look for patterns

This heuristic can guide a problem solver to successfully solve a problem where there exist some patterns in the information given. By identifying these patterns, a number of conclusions can be made. These conclusions will lead the problem solver to a solution. Consider the following example: Given (1,2), (3,4,5), (6,7,8,9), (10,11,12,13,14), … Find the numbers in the 8^{th} bracket.

First, we identify the pattern formed by the first numbers of all these brackets. These first numbers form the sequence: 1, 3, 6, 10, … = $\frac{1\times 2}{2}, \frac{2\times 3}{2}, \frac{3\times 4}{2}, \frac{4\times 5}{2}, \ldots$

The conclusion is that the first number in the n^{th} bracket = $\frac{n(n+1)}{2}$.

$\therefore$ the first number in the 8^{th} bracket = $\frac{8(8+1)}{2}$ = 36

Second, we identify the pattern formed by the number of numbers in each bracket:

2, 3, 4, 5, … = 1+1, 2+1, 3+1, 4+1, …

The conclusion is that the number of numbers in the n^{th} bracket = n+1.
$\therefore$ the number of numbers in the 8^{th} bracket = 9.
$\therefore$ the numbers in the 8^{th} bracket are (36, 37, 38, 39, 40, 41, 42, 43, 44).

Try a simpler problem

A difficult problem can be broken up into a number of simpler problems. We solve these simpler problems first. Then we use these results to obtain a solution of the original problem.

For instance, consider the following problem: Prove that for all sets of real numbers a, b, c and d, $a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da$ implies that a = b = c = d.

First, we take a simpler problem with fewer variables. That is: Prove that if a and b are real numbers, $a^2 + b^2 = 2ab$ implies a = b. For this simpler problem, it can be easily observed that $a^2 + b^2 = 2ab$ implies that $(a – b)^2 = 0$. This gives the result a = b.

The original problem when multiplied by 2 gives: $2a^2 + 2b^2 + 2c^2 + 2d^2 = 2ab + 2bc + 2cd + 2da$. This can be rearranged to give: $(a^2 + b^2) + (b^2 + c^2) + (c^2 + d^2) + (d^2 + a^2) = 2ab + 2bc + 2cd + 2da$ which yields $(a - b)^2 + (b - c)^2 + (c - d)^2 + (d - a)^2 = 0$. By making use of the above simpler problem, it follows that a = b, b = c, c = d and d = a. This gives the result a = b = c = d.

Work backwards
This heuristics is useful to solve problems to find and problems to prove. After clarifying the given information and the goal of a problem, a problem solver starts from the goal. Consider the following problem: Explain how to get 60 cm^3 of water by using two containers that can measure 40 cm^3 and 90 cm^3. If we want to have 60 cm^3 in the larger container, one must pour away 30 cm^3. This can be done if the smaller container has 10 cm^3 of water in it. How can one have 10 cm^3 in the smaller container? First, the larger container is filled to full capacity. This is followed by pouring 40 cm^3 into the smaller container and throwing it away, twice in succession. Finally, the remaining water from the larger container poured into the smaller container is of a volume of 10 cm^3.

The importance of heuristics

Pólya maintains that heuristics can be taught. He devotes much of his effort to showing how heuristic reasoning can be taught through heuristic methods. Wilson (1967) suggests that there is a need to distinguish between the heuristic method as a way of teaching and the use of heuristic reasoning as a goal. The former is to help students to tackle the problem at hand, whereas the latter is to develop students' problem-solving ability when tackling complex, non-routine problems. Pólya (1973) clearly states these goals as he comments: 'There are two aims which the teacher may have in view when addressing to his students a question or a suggestion …: First, to help the student to solve the problem at hand. Second, to develop the student's ability so that he may solve future problems by himself.' The teaching through heuristic method can be further enhanced: '... when the teacher solves a problem before the class, he should dramatize his ideas a little and he should put to himself the same questions which he uses when helping the students.' (p. 3, 4, 5)

It is hoped that there is a transfer of skills and transfer of responsibility from the teacher to the students from such a teaching approach. The students will eventually assimilate Pólya's list of questions and suggestions and start putting to themselves the same questions and suggestions, which will bring about success in tackling problems. Teachers have to be careful in the ways they put forward questions or suggestions to the students. Questions should be posed whether or not students are on the right track, to avoid students misinterpreting them. For instance, when we direct the question 'can you check your solution?' to a student, the student's first impression will be that the solution is wrong. So, instead of 'looking back' on the solution, the student may start to find a new solution.

Schoenfeld supports the ideas on heuristics put forward by Pólya. He has carried out a number of studies on heuristics from the 1980s. Schoenfeld (1985) gives a rationale for the study of heuristics and for teaching problem solving through heuristics:

(1) Our students will develop a personal and idiosyncratic collection of problem solving strategies, which can be called upon when tackling new problems.
(2) There is a substantial degree of homogeneity in ways that expert problem solvers approach new problems. It is worthwhile to make explicit these ways to our students.
(3) By means of introspection or by making systematic observations of experts solving a large number of problems, it might be possible to identify and characterize the heuristic strategies that are used by expert problem solvers.
(4) By providing direct instruction on these heuristics, our students save time in discovering them by themselves.

METHODOLOGY

This research is predominantly quantitative. It consists of two phases, in attempting to gather information to answer the objectives. The objectives are to identify the ability of form four students employing heuristics, to examine the consistency of these students employing heuristics, to examine the impact of heuristics teaching on students' mathematics problem solving and to recommend a guideline and the list of heuristics to be used for each mathematics topics. **Phase I –** The participating students will be given a pre-test. **Phase II –** The participating students will be given 3 post-tests after an intervention on heuristics.

Sample of the study

Multi-stage sampling method was used to select the required sample of study. First of all, 40 secondary schools were selected from the 154 secondary schools in Sarawak. This was followed by selecting randomly one Form Four Science class and one Form Four Art class from each of the participating schools. All the students from these selected classes became the sample of this study. The teachers teaching the selected classes from all the participating schools also became another sample of this study.

Instrumentations

For phase I of this research, the researchers designed 1 set of pre-test question papers for students aiming to gather information on their ability in employing heuristics while solving mathematics problems. For phase II, the researchers designed one set of heuristics module for intervention in the classrooms by the participating teachers and 3 sets of post-test question papers aiming to gather students' ability in employing heuristics after the intervention. On top of these, two sets of questionnaires were also designed and to be given to the participating teachers and students aiming to gather their demographic information, their problem-solving skills and their perceptions of heuristics.

Research design

Figure 1 shows the research design intended for this project.

Data analysis

The data collected will be analyzed using SPSS version 12.01. Frequency distribution, mean, factor analysis, t-test, ANOVA and regression analysis will be used to solicit evidence to support the argumentations for the various objectives of this project.

PILOT TEST FINDINGS

Up to this point in time, we have completed the pilot test on the instruments of this study. This part presents some of the findings of the pilot test.

Heuristics and overall achievement

Students participating in the pilot test were evaluated based on their use of heuristics and the overall scores in answering the pre-test question paper which was made up of 4 open-ended questions and 4 guided questions. Table 1 shows the result of the analysis done on the 4 open-ended questions. A total of 159 attempts by students used heuristics. Out of this number, 63.5% or 101 questions had the correct answers. However, only 22 out of the 77 attempts by students or 28.6% obtained the correct answers without using heuristics. This implies that heuristics is important to successful mathematics problem solving.

Table 2 shows the result of the analysis on students’ attempts on the guided questions. A total of 228 attempts using heuristics were recorded. The increase in the number of attempts using heuristics subsequently resulted in the increase in percentage of obtaining correct answers. This is proven by 78.9% for guided questions against 63.5% for open-ended questions of successful attempts in using heuristics. This finding further supports the fact that heuristics does play an important role in mathematics problem solving.

Further analysis shows that there is a high correlation ($r = 0.683$) between the use of heuristics and the overall score.

Heuristics and problem-solving skills

Table 3 shows the result of the analysis on students’ responses on the set of questionnaire. The high correlation between devising a plan and the use of heuristics (0.457) indicates the importance of employing heuristics while planning a solution for a mathematics problem.

CONCLUSIONS

Teachers should illustrate the importance of heuristics or strategies to students in problem-solving activities. By doing so, heuristics will exist as a stock of tools which students can readily call into play at any time when rules and algorithms fail to tackle a problem.

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APPENDICES

Table 1: Distribution of Students' Use of Heuristics and Their Scores for Open Ended Questions

Question	Use of Heuristics			
	Yes		No	
	Correct Answer	Wrong Answer	Correct Answer	Wrong Answer
1	36 (76.6%)	11 (23.4%)	3 (25.0%)	9 (75.0%)
2	19 (67.9%)	9 (32.1%)	13 (41.9%)	18 (58.1%)
3	28 (77.8%)	8 (22.2%)	3 (13.0%)	20 (87.0%)
4	18 (37.5%)	30 (62.5%)	3 (27.3%)	8 (72.7%)
Total	101 (63.5%)	58 (36.5%)	22 (28.6%)	55 (71.4%)

Table 2: Distribution of Students' Use of Heuristics and Their Scores for Guided Questions

Question	Use of Heuristics			
	Yes		No	
	Correct Answer	Wrong Answer	Correct Answer	Wrong Answer
1	48 (85.7%)	8 (14.3%)	3 (100%)	0 (0%)
2	53 (91.4%)	5 (8.4%)	1 (100%)	0 (0%)
3	48 (82.8%)	10 (17.2%)	0 (0%)	1 (100%)
4	31 (55.4%)	25 (44.6%)	1 (33.3%)	2 (66.7%)
Total	180 (78.9%)	48 (21.1%)	5 (62.5%)	3 (37.5%)

Table 3: Spearman Correlation between Problem Solving Skill, Heuristics

Problem-solving skills	Heuristics
Understanding the problem	0.104
Devising a plan	0.457
Carrying out the plan	0.211
Looking back	0.277

Figure 1: Research design