Primary Five Pupils' Solution Strategies, Modes of Representation, Justifications and Errors in Solving Pre Algebra Problems

by

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ABSTRACT

This case study aimed at looking into how Primary 5 pupils solve pre algebra problems concerning patterns and unknown quantities. Specifically, objectives of this study were to describe Primary 5 pupils' solution strategies, modes of representations, justifications and errors in: (a) discovering, describing and using numerical and geometrical patterns, and (b) solving for unknown quantities in word problems. Subjects of this study consisted of four Primary 5 pupils from a primary school in Kota Samarahan, Sarawak. The data of this study consisted of verbal and written data. All the four pupils were given a set of five pre algebra problems. The first problem concerned numerical pattern while the second and third problems concerned geometrical patterns. The fourth and fifth problems were word problems involving unknown quantities. Pupils were asked to write down all the steps used in solving those problems and at the same time verbalize their thinking. Data were collected through pupils' verbal think aloud protocols, retrospective questioning and observation. All verbal data were transcribed before analyzed. All written work by pupils was also analyzed. Findings of this study seemed to suggest that pupils displayed different solution strategies and used various modes of representation to solve problems concerning patterns. In solving for unknown quantities in word problems, pupils justified their strategies differently even though their solution strategies appeared similar. Errors made in the process of solving those pre algebra problems were also discussed.

BACKGROUND OF THE STUDY

Fearnley-Sander (2000) pointed out that "interest in algebra education for students at an early age is recent and so there are as yet only a few studies in this area" (p. 85). This statement may be related to some "history" of early algebra education.

In the 80s, National Council of Teachers of Mathematics (NCTM) called for focusing algebra across the grades, begun as early as preschool. Consequently in 1989, NCTM's Curriculum and Evaluation Standards for School Mathematics promoted algebra as a K-12 enterprise (Moses, 1997). So finally in 1994, the Algebra Working Group appointed by NCTM introduced the emerging view of algebra which acknowledged the "dynamic nature of mathematics in general and of algebra in particular, treats mathematics as a human activity" (Davis and Harsh, 1981, cited in Yackel, 1997, p.276). This group advocated that all children can learn algebra.

Plager, Klinger and Rooney (1997) pointed out that elementary school children "display a tremendous intellectual curiosity about number patterns" (p.330). They therefore suggested that children can be encouraged to engage in algebraic activities such as recognize, describe, extend and create a wide variety of patterns. Besides, there were also a number of studies undertaken by overseas researchers on preschool and elementary school students' abilities in solving algebra tasks and problems (e.g. Curcio and Schwartz, 1997; Lubinski and Otto, 1997; Land and Becher, 1997; Cai, 1998).

However, in Malaysia, algebra may seem to be a very strange word in the minds of the primary school pupils. This is not surprising at all since algebra is most probably never been taught formally and directly to them in the classroom. In fact, mathematics under the New Primary School Curriculum (*Kurikulum Baru Sekolah Rendah* or *KBSR*) actually "contains" some elements of algebra. For instance, finding missing addend, minuend or subtrahend in arithmetic equations is actually algebraic as it involves the process of organizing the arithmetic needed to find an answer to a question involving quantities that are not yet known. Choike (2000) defined this process as "algebra".

STATEMENT OF THE PROBLEM

This study was thus undertaken to look into the abilities of Primary 5 pupils in solving pre algebra problems based on their prior knowledge and experiences in mathematics. Considering that they are not exposed to algebra directly and formally in the classroom, how would they solve pre algebra problems in terms of their solution strategies and modes of representation used? How would they justify their solution processes and what could be the possible types of errors made while solving pre algebra problems?

PURPOSE OF THE STUDY

This study aimed at looking into how Primary 5 pupils solve pre algebra problems. For the purpose of this study, the scope of pre algebra problems were limited to those involving (a) numerical and geometrical patterns, and (b) unknown quantities.

Specifically this study intended to describe how Primary 5 pupils discover, describe and use numerical and geometrical patterns. These "growing patterns" required pupils not only to extend patterns but also look for a generalization or an algebraic relationship that will tell them what the pattern will be at any point along the way. For problems involving unknown quantities, this study would describe how Primary 5 pupils use mathematical operations to solve for unknown quantities presented in the form of word problems. Their process of solving those pre algebra problems were studied with respect to their solution strategies, modes of representation, justifications and errors.

RESEARCH QUESTIONS

In line with the objectives, this study intended to answer the following research questions:

a) What are the solution strategies used by Primary 5 pupils to
   i) discover, describe and use numerical patterns?
   ii) discover, describe and use geometrical patterns?
   iii) solve for unknown quantities in word problems?

b) What are the modes of representation used by Primary 5 pupils to solve pre algebra problems concerning
   i) numerical patterns?
   ii) geometrical patterns?
   iii) unknown quantities in word problems?

c) What are the justifications used by Primary 5 pupils in solving pre algebra problems concerning
   i) numerical patterns?
   ii) geometrical patterns?
   iii) unknown quantities in word problems?

d) What are the errors made by Primary 5 pupils in solving pre algebra problems concerning
   i) numerical patterns?
   ii) geometrical patterns?
   iii) unknown quantities in word problems?

SOME RELATED LITERATURE REVIEW

Pre Algebra

Friedlander and Hershkowitz (1997) suggested that pre algebra should cover an understanding of variable, expression, function and equation, as well as the ability to construct and analyze multiple representations of number patterns and situations. They also added that the process of generalizing and justifying patterns at the level of pre algebra require the students to produce some additional examples of the 'same kind'; employ the evolving pattern in some given situation; generalize the pattern and justify conclusions.

Urquhart (2000) (p.1) also suggested that some algebraic skills could be developed early. Among such skills include (a) recognizing patterns and functions; (b) using pictures, graphs, tables and equations to represent relationships; (c) understanding numbering systems; (d) working with properties of operations; and (e) using variables and open structures to represent quantities and express relationships.

To summarize, pre algebra concerns recognizing, generalizing and justifying patterns which involves constructing various representations. It also involves understanding expressions, equations and number system in order to work with unknowns or variables and properties of operations. Therefore, the pre algebra problems used in this study focused on (a) recognizing, generalizing and justifying patterns, and (b) working with unknowns in the form of word problems.

Operation Sense

Slavit (1999) explained that the ability to use the operation as “operation sense”. He then further elaborated that “operation sense involves various kinds of flexible conceptions” (p.254) about the underlying structure and use of mathematical operations as well as relationships among these operations.

Slavit (1999) pointed out that “early elementary school students are capable of making sense of advanced notions of arithmetic that transcend into algebraic realms” (p. 251-252). He also elaborated that “students at the age of 6 and 7 are quite capable of developing deep understandings of mathematical processes and can be well on their way to developing algebraic ways of thinking” (p.273).

Schifter (1999) also explained that when the children come to see that any missing addend problem can be solved by subtraction, they evidence a sense of how the operations are related and acquired experience with the inverse relationships of addition and subtraction. This is related to MacGregor and Stacey’s (1999) view that ability to see the reasons behind relationships requires a generalization about properties of numbers and this ability is deeply algebraic.

Solving Algebra Word Problems

Cai (1998) concluded from his study that fourth- and sixth-grade students were “able to use algebraic approaches to solve problems” (p.226). In another study, Palomares and Hernandez (2002) claimed that fifth graders used informal arithmetic strategies or non-school strategies to solve those algebra word problems. In their study involving fifth graders, they found some strategies systematically used by students during the experimental phase – propose a number and check it to find a solution; base their work on the design of a drawing to find the solution; draw a number line to compare paths covered by a series of jumps; mechanical use of basic arithmetic operations; and, preference for the use of mental arithmetic without having to write the operations used (numeric answer).

Cognitive Analysis of Problem Solving

Charles & Silver (1988) and Silver (1987) both pointed out that there four cognitive aspects that are important and significant dimensions in mathematical problem solving, namely solution strategies, modes of representation, mathematical justifications and mathematical errors. The four cognitive aspects concerned were discussed as follows.

Solution strategy refers to the plan used by pupils to achieve the goal of the problem. According to Anderson (1987), cognitive psychologists distinguished two types of cognitive strategies. The first type is general cognitive strategies for problem solving such as brainstorming, means-end analysis, reasoning through analogy, the use of worked examples, working backward and working forward. These strategies can be applied to problems in many different domains. The second type of cognitive strategy is domain-specific strategies such as looking for a pattern, which may only be applied to problems in a particular domain such as mathematics, particularly algebra.

Glaser (1987) explained that proficiency in mathematics problem solving depended on the acquisition, selection and application of both general problem solving strategies and domain or content specific strategies. Thus, use of different strategies reflected individual differences in mathematics problem solving. This implied that examination of the strategies used can provide information regarding pupils' thinking and reasoning.

Modes of representation are the external representations of students' solution processes which reflect their mathematical thinking (Cai, 1995). Cai (1995) classified modes of representation into verbal (spoken or written words), visual (picture or drawing), arithmetic symbolic (use of numbers) and algebraic symbolic representations. Since pre algebra also concerns constructing various representations, Friedlander & Tabach (2001) introduced four types of representations in algebra context – verbal, numerical, graphical and algebraic representations. Examination of these modes of representation revealed the ways in which pupils solve problems and reflected the ways in which pupils communicate their mathematical ideas and thinking processes.

Mathematical justification is related to communication. In solving mathematical problems, pupils could be asked to justify their answers and solution processes, make and evaluate mathematical conjectures and arguments and validate their own thinking. McCoy, Baker & Little (1996) stressed that students actually seek understanding when they conjecture, argue and justify. To evaluate the quality of justification, Voss, Perkins & Segal (1991) proposed that mathematical justifications are judged in terms of their soundness. Soundness refers to whether the justification providing support is (a) acceptable or correct and (b) complete.

Mathematical error "is a natural part of mathematics reasoning" (Bruning, Schraw & Ronning, 1995, p.340). Errors might reflect pupils' lack of conceptual, procedural or metacognitive knowledge. Previous studies have demonstrated the value of error analysis in capturing pupils' understanding of mathematical knowledge. The examination of pupils' misconceptions or errors provided an indication of pupils' levels of proficiencies in mathematics problem solving, thinking and reasoning.

LIMITATIONS OF THE STUDY

This study made use of techniques such as collecting and analyzing verbal think aloud protocols during task-based interviews. According to Cai (1995), "the process of collecting, coding and analyzing verbal protocol data is extremely labour intensive" (p.7). Therefore, a large sample is not feasible for this study. Consequently, only four Primary 5 pupils were involved in this study. Thus, the results of this study were indicative and could only be used only to describe the pattern in the sample.

METHODOLOGY

Design of the Study

This descriptive cognitive study took the design of a single site case study as only one primary school was involved in this study.

Subjects of the Study

A number of studies had indicated ability of 6^{th} Grade students in using algebraic symbolism (Land & Becher, 1997) and to think algebraically (Cai, 1998). However in Malaysia, Primary 6 is the year where the pupils are required to sit for a public examination towards the end of third quarter of the year. In order to avoid possible disruptions towards their learning process in the classroom, Primary 5 pupils were chosen instead. This could be justified as there were also studies indicating ability of 4^{th} Grade students to reason algebraically if given the opportunity (Lubinski & Otto, 1997).

Consequently, the subjects of this study consisted of four Primary 5 pupils from one primary school in Samarahan Division of Sarawak. The selection of subjects were based on one criterion that the subjects need to be able to articulate verbally well due to the data collection method chosen for this study.

Data Collection

Since this study involved knowledge elicitation, techniques like process tracing through "talk- or think-aloud" method became the main means of data collection. This way of verbal protocol data collection involved presenting the problems to subjects along with verbalization instructions. Collection of verbal think aloud protocols yielded information about the knowledge and thought processes that underlie observable task performance (Chipman, Schraagen, & Shalin, 2000). In this way, the thought processes underlying subjects' solution processes and justifications could be collected through these verbal think aloud protocols. Indirectly, their errors might be detected as well.

Verbal protocol was carried out concurrently and retrospectively. Concurrent protocol was done by asking subjects to solve the pre algebra problems and at the same time asking them to verbalize their thinking. According to Ericsson and Simon (1980,1984) (cited in Hassebrock & Prietula, 1992), concurrent verbalization provided the most complete report since information was verbalized as processing and verbalization occurred at the same time and therefore, no thought, feeling, or action would be omitted because the participant had no time to forget! This added to the validity of this method in collecting data about thinking processes. Sometimes, retrospective questioning was done after concurrent protocol as a supplement to provide the missing information or to fill the gaps in concurrent protocol.

Once the verbal protocols were collected on audio tape, they must be transcribed, segmented into codable units of subject statements, coded according to a coding scheme and analyzed to describe the cognitive processes subjects used to solve pre algebra problems.

Task-based interview was used in this study to support the verbal think aloud protocol method. Task-based interview is actually a research instrument for making systematic observations in the psychology of learning mathematics and can be adapted as assessment tools for describing the subject's knowledge (Goldin, 2000). It focuses research attention more directly on the subject's process of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results he/she produces.

In addition to that, a more detailed description of the kinds of strategies used by students can also be obtained through observation (Cai, 2001). The primary focus of observation in this study was the subjects' order of solution processes and modes of representations used in solving the pre algebra problems. Indirectly, subjects' errors in their written solution processes and modes of representation could also be observed.

Instrument

A set of five pre algebra problems were administered to the subjects to explore their strategies used. These problems allowed the subjects to produce and display their process used to obtain an answer. This provided visible record of their solution processes and use of representations.

These five pre algebra problems were taken and adapted from a few studies (e.g. Kaput & Blanton, 2001; Femiano, 2003; and Larkin, Perez, & Webb, 2003). Problem 1 required subjects to discover, describe and use numerical pattern whereas Problems 2 and 3 concerned that of geometrical patterns. Problems 4 and 5 required the subjects to solve for the unknown quantities in the word problems. These problems were used as they covered the scope of pre algebra as discussed in literature review.

For the purpose of this study, these problems were translated by the researcher into the Malay language which is the medium of instruction used in Primary 5 classroom in the teaching and learning of mathematics.

Execution of the Study

In consideration that the main source of data for this study came from subjects' verbal data, thus rapport with the subjects was very important to elicit data from them. To achieve this, the subjects were introduced to the researcher through the school teacher. The researcher emphasized to all the subjects that the solution process, not the answer, was more important in this study. Assurance of confidentiality and anonymity was also stressed to every subject before the collection of verbal think aloud protocols.

Analysis of Data

The first step in analyzing a verbal protocol was to break down the transcript into short segments or phrases that could be coded with a pre-defined coding scheme. This step in analyzing the protocol yielded a topic representation in which each segment addressed a particular instance of reasoning behavior on the task. The topic representations were then coded. The coding scheme used in this study was adapted from Cai's (1995) study which focused on four main cognitive aspects of pupils' thinking and reasoning: solution strategies, modes of representation, mathematical justifications and mathematical errors.

Besides verbal protocol analysis, documentary and content analysis will be conducted. Content analysis involves identification and classification of content (Anderson, 1998). It was used in this study to describe the written solution process and modes of representation used by subjects as well as the errors made in the solution processes.

ANALYSIS OF FINDINGS

Problem 1

Apakah nombor yang sepatutnya diisi dalam petak mengikut turutan nombor berikut:
What is the number to be filled in the box following the number sequence below:

87, 81, 75, 69, ☐

Solution strategy

All the subjects seemed to discover the pattern in the numerical sequence by finding the common difference between two consecutive numbers in the sequence. Three methods were identified in finding the common difference. Two subjects used subtraction in standard algorithm, one used counting down method and one used mental subtraction method by finding the number to be subtracted from 87 to yield 81 (Refer to Diagram 1). All of them described the given numerical pattern as "descending". To arrive at the numerical answer, three of the subjects used separate standard algorithms (Methods 1, 2 and 4 in Diagram 1) whereas one of them used continuous standard algorithms (Method 3 in Diagram 1).

Mode of representation

Only arithmetic symbolic / numerical representation was used to solve this problem and verify the answer (Refer to Diagram 1).

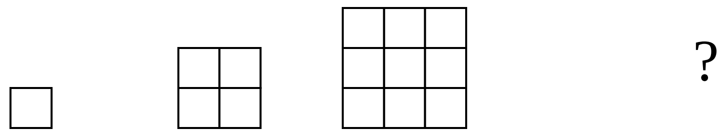
Justification

All the subjects were able to justify the process of determining the common difference and the required answer. They seemed to relate the subtraction operation to the decreasing value of terms in the numerical pattern. The subject who found the common difference once only verified his answer by confirming whether the difference is the same as the common difference for the numerical pattern (Method 1 in Diagram 1).

Errors

No major error could be identified. One subject carelessly used addition to find the answer, but changed to subtraction upon realization of the decreasing numbers in that numerical pattern (Method 2 in Diagram 1).

Problem 2



Rajah 1 Rajah 2 Rajah 3 Rajah 5

Dengan merujuk kepada Rajah 1, Rajah 2 dan Rajah 3, lukiskan Rajah 5.
By referring to Figure 1, Figure 2 and Figure 3, draw Figure 5.

Solution strategy

Three of the subjects seemed to discover, and then described the geometrical pattern in terms of the number of rows and columns. Another one subject saw and described the next figure as construction from previous figure (Refer to Diagrams 2a and 2b).

Among the three subjects who interpreted Figures 1, 2 and 3 in terms of number of rows and columns, one of them immediately drew a big square with 16 squares in 4 rows and 4 columns as Figure 5. After being asked to justify his answer, he drew another big square with 25 squares in 5 rows and 5 columns and called it Figure 5. Another two subjects each drew a big square with 16 squares in 4 rows and 4 columns and named it as Figure 4. Then they continued to draw another big square with 25 squares in 5 rows and 5 columns and called it Figure 5.

Another subject solved this problem by constructing Figure 4 based on Figure 3 by adding 7 squares in an inverted “L” shape, thus yielding a bigger square with 16 squares in 4 rows and 4 columns (Refer to Diagram 2c). In the similar way, he constructed Figure 5 based on Figure 4 (Refer to Diagram 2d).

Mode of representation

Three subjects used verbal and visual representations to solve this problem but justified their solution verbally based on given diagrams. Another subject used only visual representation to get the answer but used visual and verbal representations to justify his answer (Refer to Diagram 2).

Justification

Justification was based on the increasing number of rows and columns. When asked to justify their answer, three subjects generalized that "Figure 1 has 1 row and 1 column; Figure 2 has 2 rows and 2 columns; Figure 3 has 3 rows and 3 columns; Therefore, Figure 5 must have 5 rows and 5 columns". Another justification was based on the "construction" method. One subject explained and drew how Figure 2 was constructed from Figure 1 (Refer to Diagram 2a). He then explained and drew how Figure 2 was constructed to form Figure 3 (Refer to Diagram 2b).

Errors

Two major errors were found. First, generalizing number of squares added from Figure 1 to Figure 2, and applied on Figure 3 to get the next figure, without confirming the number of squares added from Figure 2 to Figure 3. The second error was the ignorance of Figure 4 which existed before Figure 5.

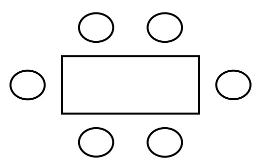
Three subjects who interpreted Figures 1, 2 and 3 in terms of number of squares concluded immediately that there were 12 squares, without specifying whether it was Figure 4 or Figure 5. Two of them were unable to justify their answer. Another one explained that "3 squares were added to Figure 1 to yield Figure 2". Thus she explained that "3 squares were also added to 9 squares in Figure 3 to yield 12 squares". Her explanation seemed to suggest that she did not confirm the pattern by comparing Figure 2 and Figure 3 and that she seemed did not realize the existence of Figure 4 before Figure 5! All these three subjects also did not give the final answer in terms of number of squares in Figure 5, as required by the question.

Another subject who used the "construction" method concluded that the number of squares in Figure 5 was 23! According to him, 7 squares were added to Figure 3 (which has 9 squares) to yield 16 squares in Figure 4. Thus, 7 squares were added to Figure 4 to yield 23 squares in Figure 5. But when he was asked to justify his answer, he referred to the Figure 5 which he has constructed, counted all the squares systematically and changed his final answer to 25. Does this suggest that the subject was facing the difficulty in connecting the two types of representations used, namely visual and arithmetic symbolic representations, or it could be due to simply carelessness?

Problem 3

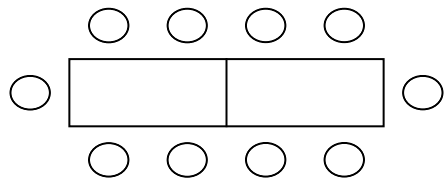
Ali sedang mengatur meja untuk suatu majlis yang meraikan harijadinya. 6 orang boleh duduk mengelilingi sebuah meja seperti berikut:

Ali is arranging table for his birthday party. 6 persons can be seated around the table as below:



Apabila Ali menyambung dua buah meja seperti berikut, 10 orang boleh duduk mengelilingi meja tersebut seperti berikut:

When Ali puts two tables as below, 10 persons can be seated around the table as below:



Berapakah orang yang boleh duduk mengelilingi 3 buah meja yang disambung secara hujung ke hujung?

How many persons can be seated around 3 tables which are put end to end?

Solution strategy

Two subjects seemed to discover the pattern by comparing the first and second diagrams. They did not describe the pattern verbally but produced the required diagram almost instantaneously. Two other subjects seemed to relate the two diagrams to a numerical pattern which reflected the number of persons seated around the table. All the subjects arrived at the answer through drawing with reference from previous drawings. One subject constructed his diagram based on the diagram showing 2 tables with ten "persons" seated around it. He shifted the "person" sitting at the width of the second table to the width of the third table and put four "persons" on the lengths - two on each side of the length of the third table (Refer to Diagram 3b). The other subjects drew a diagram consisted of three tables arranged end to end, then seated 14 "persons" around the three tables (Refer to Diagram 3c) to get the answer.

Mode of representation

All the subjects used visual representation to solve the problem (Refer to Diagram 3). However, two of them used verbal and numerical representation to verify their answer.

Justification

The justification was based on extension of diagram – "follow the diagram before this" as said by the subjects. Two subjects used numerical pattern with a common increment in value to verify their answer. One of the subjects justified her verification by pointing at the diagram with one table and said "six", then she pointed at the diagram with two tables arranged end to end, she said "ten" and mentioned "six plus four is ten". Finally she said "so ten plus four is fourteen".

Errors

One subject made a wild guess of 15 but was unable to justify her answer. Then she drew a diagram consisted of three tables arranged end to end. She faced the problem of where to locate the "person" who was previously seated at the end of second table. She put that "person" on the length of the first table (Refer to Diagram 3a) but soon realized her mistake when she referred to the diagram with one table and six people seated around it.

Problem 4

> Bahtiar telah membaca beberapa buah buku. Jika dia membaca 5 buah buku lagi, jumlah buku yang dia baca akan menjadi 17 buah buku. Berapakah buah buku yang telah dibaca oleh Bahtiar sebelum ini?
> *Bahtiar has read some books. If he reads 5 more books, the total number of books he reads will become 17 books. How many books has Bahtiar read before this?*

Solution strategy

All the subjects used subtraction operation to get the answer. Some examples of subjects' verbal protocols that suggested their line of thought were: "number of books read is 17, including 5 more books" and "he needs 5 more books to make up 17". Analysis of these verbal protocols seemed to suggest they were thinking along "? + 5 = 17" to solve this problem.

Mode of representation

Only arithmetic symbolic / numerical representation was used to solve this problem and verify the answer (Refer to Diagram 4).

Justification

Subtraction operation was justified differently. One subject based on the word "before this" ("*sebelum ini*"). Another subject said "because he needs 5 more books to make it 17, so 17 minus 5". Two subjects said "12 plus 5 is 17, so I used 17 minus 5 to get 12". These two subjects verified their answer by using addition (Method 2 in Diagram 4).

Errors

No error was found.

Problem 5

Mariam akan berumur 20 tahun dalam masa 3 tahun lagi. Umur abangnya, Dahlan adalah 2 tahun lebih daripada umur Mariam. Berapakah umur Dahlan sekarang?

Mariam will be 20 years old in 3 years. Her brother, Dahlan's age is 2 years more than Mariam. What is Dahlan's age now?

Solution strategy

Three subjects used subtraction and addition operations while another one used counting back and then counting on to solve the problem. Two subjects used subtraction to obtain Mariam's age and used addition to get the answer (Refer to Method 1 in Diagram 5). One used mental subtraction to get Mariam's age and wrote "17 + 2 = 19" in standard algorithm as the answer. Another one used the counting back method to get Mariam's age and then counting on in finding Dahlan's age (Refer to Method 2 in diagram 5).

Mode of representation

One subject used a combination of verbal and arithmetic symbolic / numerical representation whereas the others used only arithmetic symbolic / numerical representation.

Justification

The subtraction operation and counting back method was justified to get the present value due to the phrase "in 3 years" whereas the addition operation and counting on method was justified through being "2 years more".

Errors

One of the subjects used addition operation (Refer to Diagram 5a) but could not justify his answer. After reading the problem again, he managed to solve the problem.

DISCUSSION OF THE FINDINGS

Numerical Pattern

Subjects seemed to be able to discover the regular common difference from one number to the previous or next one in the numerical pattern. They were able to describe the trend of numerical pattern and then generate the following number from the previous number based on the common difference identified in the number pattern. Verification of the required answer was done by using the regular common difference. Subjects tended to use numerical representation only in discovering, describing and extending numerical pattern. No major error was found among the subjects in dealing with numerical pattern presented in this problem.

Geometrical Patterns

For the problem involving irregular or growing geometrical pattern (Problem 2), the subjects seemed to discover and describe the pattern differently. Three of the subjects who described the pattern in terms of number of rows and columns had the difficulty in generating the required figure. Moreover, the problem required them to generate the fifth figure based on first, second and third figures. Their generalization was incomplete and they overlooked the existence of the fourth figure. In addition, these three subjects did not mention the required numerical answer. However, they were still able to justify their solution process verbally with the aid of diagrams.

One subject exhibited his ability to discover and describe the next figure as being the construction of the previous figure. He constructed the required figure without any difficulty and was able to justify his solution process correctly. Could this be due to the way he discovered and described the pattern in a "constructive" method that led him to the answer easily compared to the other subjects?

Problem 3 involved a regular geometrical pattern. All subjects were able to recognize the pattern "hidden" in the diagrams though the time taken ranged from instantaneously to pause for 30 seconds. Two subjects seemed to be able to relate the visual representation used in the problem to numerical representation in justifying their answer.

Unknown Quantities in Word Problems

Problem 4 was a single-step problem involving missing addend as interpreted by the subjects. Two subjects' made their efforts to verify their answer. Using subtraction operation to find the missing addend and then used addition operation to verify the subtraction operation reflected the subjects' ability of "operation sense" as mentioned by Slavit (1999). These two subjects seemed to be able to understand and apply the inverse relationships between addition and subtraction operations. This finding also seemed to be in line with what Schifter (1999) said – when the children come to see that any missing addend problem can be solved by subtraction, they evidenced a sense of how the operations are related and acquired experience with the inverse relationships of addition and subtraction. The subjects were able to justify their use of operations correctly and completely.

Problem 5 was a multiple-step problem. Two of the subjects voiced their same confusion whether to find Mariam's age now or three years later. After reading the problem again, they were able to solve this problem. Subjects used both formal (arithmetical and numerical) and informal (counting on and counting back) methods in solving this problem. In terms of their justifications, they seemed able to explain why subtraction and addition operations, or counting back and counting on were used to achieve the subgoal and goal of the problem.

In solving Problems 4 and 5, the subjects tended to use numerical and arithmetic symbolic representation. They seemed to display some strategies as identified by Palomares and Hernandez (2002) as discussed in literature review. For instance, use of mental arithmetic without having to write the operations used and mechanical use of basic arithmetic operations which led to errors.

CONCLUSIONS

This study was undertaken to describe the solution strategies and modes of representation used, as well as the justifications and errors made by four Primary 5 pupils from a rural school while solving pre algebra problems concerning patterns and unknown quantities. Based on the analysis and discussion of findings, some conclusions could be drawn for this study.

All the subjects used domain-specific strategies particularly “look for a pattern” to solve problems concerning patterns. For word problems most subjects preferred to use formal strategies particularly arithmetical and numerical strategies involving numbers and operations. General problem solving strategies like “working forward”, “working backward”, "identifying subgoal" and “drawing diagram” were also used.

Verbal and arithmetic symbolic / numerical representation seemed to be the most commonly used modes of representation in the solution process, including verification of answers in almost all problems. Visual representation was used only in problems involving diagrams, particularly geometrical patterns. Algebraic symbolic representation was not used directly at all!

The subjects seemed like to justify their solution processes and answers verbally for pre algebra problems concerning numerical patterns and unknown quantities in word problems. For problem concerning geometrical patterns, verbal justifications were made with the aid of diagrams.

Inaccurate generalization was the major error found in problem involving irregular geometrical pattern (Problem 2). Misinterpretation and misunderstanding of problem seemed to happen for Problem 2 and Problem 4 as these two problems involved multiple-step. No major errors were found for the other problems, except carelessness and mechanical use of operations that led to incorrect answer.

IMPLICATIONS OF THIS STUDY

Findings of this study indicated that the subjects were able to solve pre algebra problems to a certain extent. They exhibited their ability to use the “look for a pattern” strategy to solve problems involving regular patterns. They took shorter time to solve problems concerning regular numerical pattern compared to regular geometrical or pictorial pattern. Could this be due to subjects' familiarity with numerical patterns compared to geometrical patterns? In addition, most subjects also seemed to face difficulty in discovering and describing irregular or ‘growing’ geometrical pattern. Again, could this be due to subjects' familiarity with regular patterns compared to irregular patterns? Could this familiarity be related to classroom's teaching and learning activities?

Some subjects displayed their ability to make connection among two different modes of representation. According to Driscoll and Moyer (2001), ability to make connections among different representations is one indicator of algebraic thinking – ability to represent, generalize and formalize patterns and regularity (Van De Walle, 2001). Some subjects also seemed to possess certain ability in “operation sense”. Slavit (1999) argued that operation sense can be transitioned into algebraic ways of thinking. Do these abilities imply subjects' emerging ability to think algebraically, as mentioned by Cai (1998)?

However, findings of this study indicated that Primary 5 pupils did not use algebraic representation in solving pre algebra problems. Does this reflect the fact that algebraic approach is yet to be intiduced in the teaching and learning of mathematics in the primary schools, particularly in problem solving?

SUGGESTIONS FOR FUTURE STUDY

One way to improve this study is to increase the number of subjects involved. With an increase in the number of subjects, probably more strategies of solving pre algebra problems by Primary 5 pupils can be described. This may add to the richness of data collected.

With more subjects, Primary 5 pupils of different abilities of achievement in mathematics may be involved. Thus, comparison can be made to see the different strategies, representations, justifications and errors made by pupils of various abilities.

Another way could be to use videotaping in the data collection process. Since verbal protocols were the main source of data for this study, videotaping can help avoid the "what I say" versus "what I do" problem that might occur (Roschelle, 2000).

APPENDICES

Diagram 1 –Solution Strategies & Arithmetic Symbolic or Numerical Representation Used by Subjects in Solving Problem 1

Method 1

75	69	69
- 69	- 6	- 63
6	63	6

Method 2

87	81	75	69	69
- 81	- 75	- 69	+ 6	- 6
6	6	6	5	63
			(Error)	

Method 3

87
- 6
81
- 6
75
- 6
69
- 6
63

Method 4

87	81	75	69
- 6	- 6	- 6	- 6
81	75	69	63

<u>Diagram 2 – Solution Strategy, Visual Representation & Justification Used by One of the Subjects in Solving Problem 2</u>

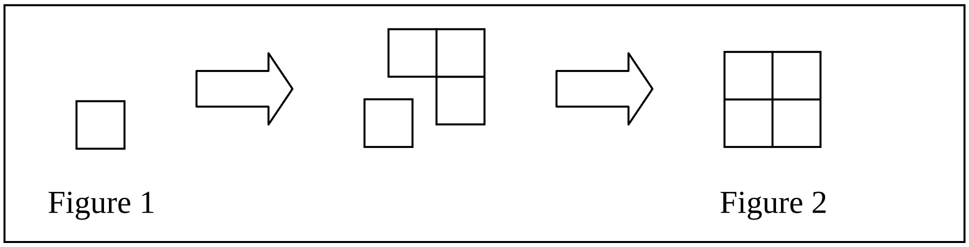


Diagram 2a – Justification on how Figure 1 transformed to Figure 2

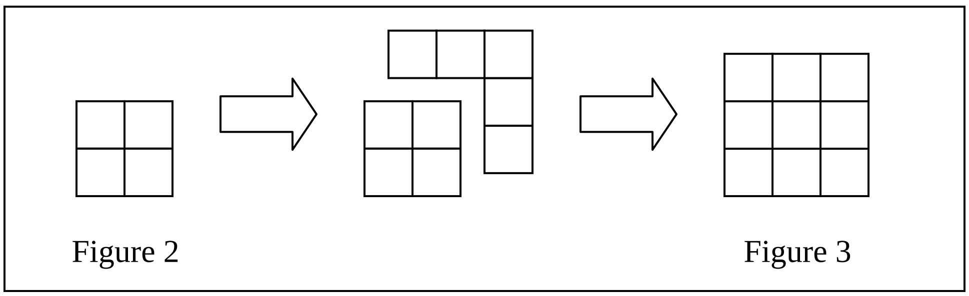


Diagram 2b – Justification on how Figure 2 transformed to Figure 3

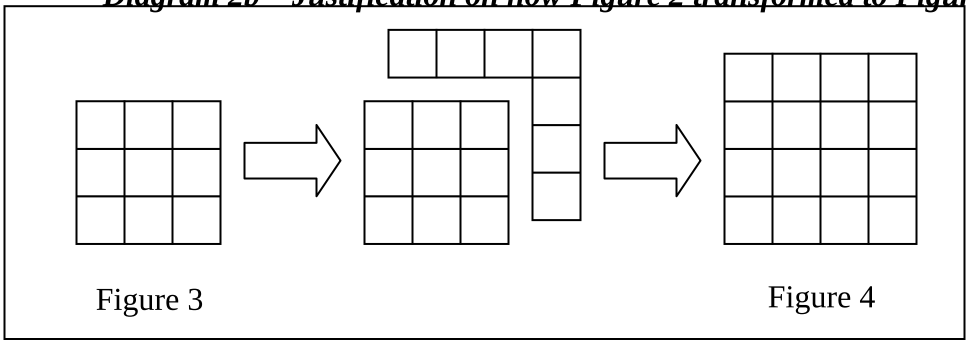


Diagram 2c – Constructing Figure 4 based on Figure 3

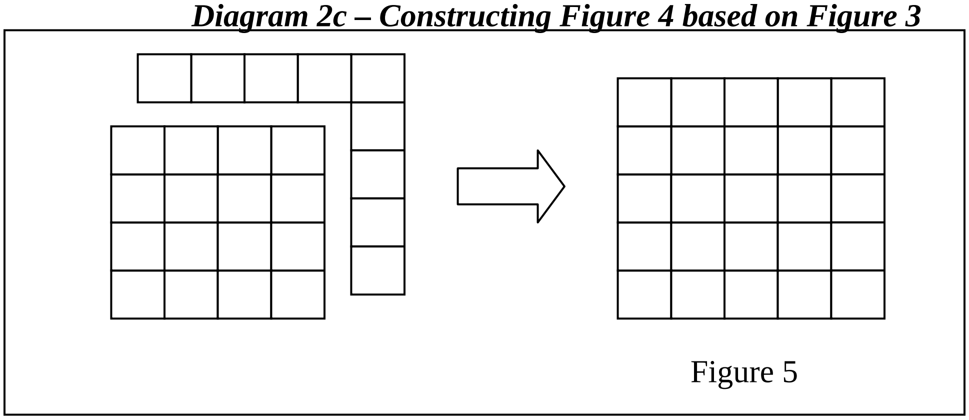
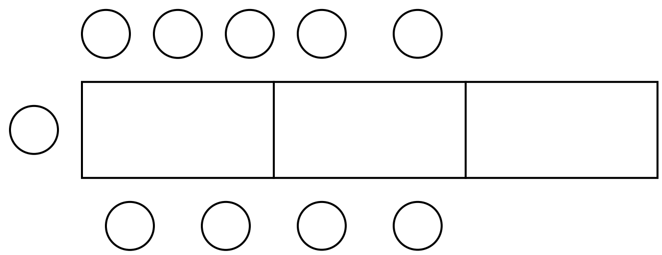


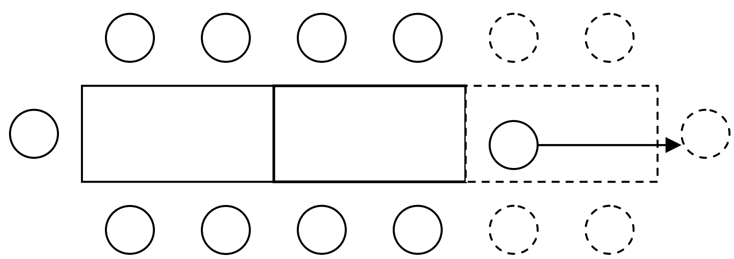
Diagram 2d – Constructing Figure 5 based on Figure 4

<u>Diagram 3 – Solution Strategies & Visual Representation Used by Subjects in Solving Problem 3</u>

<u>Diagram 3a</u>



<u>Diagram 3b</u>



Note: Solid lines represent the existing diagram in Problem 3.
Dotted lines represent extension of diagram based on existing diagram.

<u>Diagram 3c</u>

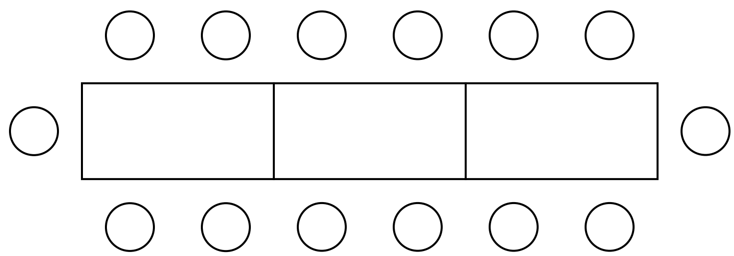


Diagram 4 – Solution Strategies & Arithmetic Symbolic or Numerical Representation Used by Subjects in Solving Problem 4

Method 1 – Solution strategy without verification

$$\begin{array}{r} 17 \\ \underline{-\ \ 5} \\ 12 \end{array}$$

Method 2 – Solution strategy with verification

$$\begin{array}{r} 17 \\ \underline{-\ \ 5} \\ 12 \end{array} \qquad \begin{array}{r} 12 \\ \underline{+\ \ 5} \\ 17 \end{array}$$

Diagram 5 – Solution Strategies Used by Subjects in Solving Problem 5

Method 1 – Formal method (Arithmetical / Numerical)

$$\begin{array}{r} 20 \\ \underline{-\ \ 3} \\ 17 \end{array} \qquad \begin{array}{r} 17 \\ \underline{+\ \ 2} \\ 19 \end{array}$$

Method 2 – Informal method (Counting back and counting on)

20 → 19 → 18 → 17
(Counting back to get Mariam's present age)

17 → 18 → 19
(Counting on to arrive at Dahlan's present age)

Diagram 5a – Error done by one Subject

$$\begin{array}{r} 20 \\ \underline{+\ \ 3} \\ 23 \end{array} \qquad \begin{array}{r} 23 \\ \underline{+\ \ 2} \\ 25 \end{array}$$

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